

# A Common Random Fixed Point Theorem for Weakly Compatible Mappings in Cone Random Metric Spaces

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**Abstract** In this paper, we prove a unique common random fixed point theorem in the framework of cone random metric spaces for four weakly random compatible mappings under strict contractive condition. Some corollaries of this theorem for three and two weakly random compatible mappings and for one random mapping are derived. Two examples to justify our theorem are given. Our results extend some previous work related to cone random metric spaces from the current existing literature.

**Keywords** Common Random Fixed Point, Cone Random Metric Spaces, Weakly Random Compatible Mappings

## 1 Introduction

Fixed point theory has the diverse applications in different branches of mathematics, statistics, engineering, and economics in dealing with the problems arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations, and others. Developments in the investigation on fixed points of non-expansive mappings, contractive mappings in different spaces like metric spaces, Banach spaces, Fuzzy metric spaces and cone metric spaces have almost been saturated. The study of random fixed point theorems was initiated by the Prague school of probabilistic in 1950's [9, 10, 23]. The introduction of randomness leads to several new questions of measurability of solutions, probabilistic and statistical aspects of random solutions. Common random fixed point theorems are stochastic generalization of classical common fixed point theorems. Random methods have revolutionized the financial markets. The survey article by Bharucha-Reid [8] in 1976 attracted the attention of several mathematicians and gave wings to the theory. The results of Špaček and Hanš in multi-valued contractive mappings was extended by Itoh [14]. Now this theory has become the full fledged research area and various ideas associated with random fixed point theory are used to give the solution of nonlinear system see [5-7, 11, 20]. Common random fixed points and random coincidence points of a pair of compatible random operators and fixed point the-

orems for contractive random operators in Polish spaces are obtained by Papageorgiou [15, 16] and Beg [3, 4].

In [12] Huang and Zhang generalized the concept of metric spaces, replacing the set of real numbers by an ordered Banach space, hence they have defined the cone metric spaces. They also described the convergence of sequences and introduced the notion of completeness in cone metric spaces. They have proved some fixed point theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. According to this concept, several other authors [1, 13, 19, 22] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space. In 2008, the assumption of normality in cone normal spaces is deleted by Rezapour and Hambarani [19], which is an important event in developing fixed point theory in cone metric spaces.

The aim of this paper is to extends the contractive condition (2.1) for four, three and two random mappings and establish a unique random fixed point results under this condition in random cone metric spaces using the concept of weakly random compatible mappings.

## 2 Preliminaries

### 2.1 Definition [21]

Let  $(E, \tau)$  be a topological vector space. A subset  $p$  of  $E$  is called a cone if the following conditions satisfied:

- (c<sub>1</sub>)  $p$  is closed, nonempty and  $p \neq \{0\}$ ;
- (c<sub>2</sub>)  $a, b \in \mathbb{R}$ ,  $a, b \geq 0$  and  $x, y \in p \Rightarrow ax + by \in p$ ;
- (c<sub>3</sub>) If  $x \in p$  and  $-x \in p \Rightarrow x = 0$ .

For a given cone  $p \subset E$ , we define a partial ordering  $\leq$  with respect to  $p$  by  $x \leq y$  iff  $y - x \in p$ . We shall write  $x < y$  to indicate that  $x \leq y$  but  $x \neq y$ , while  $x \ll y$  will stand for  $y - x \in p^\circ$ , where  $p^\circ$  indicate to the interior of  $p$ .

### 2.2 Definition [12, 24]

Let  $X$  be a nonempty set. Assume that the mapping  $d : X \times X \rightarrow E$  satisfies:

- (d<sub>1</sub>)  $0 \leq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = 0 \Leftrightarrow x = y$ ;

(d<sub>2</sub>)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;  
 (d<sub>3</sub>)  $d(x, y) \leq d(x, z) + d(z, y)$   $x, y, z \in X$ .

Then  $d$  is called a cone metric [12] or  $K$ -metric [24] on  $X$  and  $(X, d)$  is called a cone metric space [12].

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where  $E = \mathbb{R}$  and  $p = [0, +\infty)$ .

### 2.3 Example [12]

Let  $E = \mathbb{R}^2$ ,  $p = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ ,  $X = \mathbb{R}$  and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = (|x - y|, \mu(|x - y|))$  where  $\mu \geq 0$  is a constant. Then  $(X, d)$  is a cone metric space with normal cone  $p$  where  $K = 1$ .

### 2.4 Example [18]

Let  $E = l^2$ ,  $p = \{\{x_n\}_{n \geq 1} \in E^2 : x_n \geq 0, \text{ for all } n\}$ ,  $(X, \rho)$  a metric space and  $d : X \times X \rightarrow E$  defined by  $d(x, y) = \{\frac{\rho(x, y)}{2^n}\}_{n \geq 1}$ . Then  $(X, d)$  is a cone metric space.

Clearly, the above examples present that the class of cone metric spaces contains the class of metric spaces.

### 2.5 Definition [15]

Let  $(X, d)$  be a cone metric space. We say that  $\{x_n\}$  is:

- (i) a Cauchy sequence if for every  $\varepsilon$  in  $E$  with  $0 \ll \varepsilon$ , then there is an  $\mathbb{N}$  such that for all  $n, m > \mathbb{N}$ ,  $d(x_n, x_m) \ll \varepsilon$ ;
- (ii) a convergent sequence if for every  $\varepsilon$  in  $E$  with  $0 \ll \varepsilon$ , then there is an  $\mathbb{N}$  such that for all  $n > \mathbb{N}$ ,  $d(x_n, x) \ll \varepsilon$  for some fixed  $x$  in  $X$ .

A cone metric space  $X$  is said to be complete if every Cauchy sequence in  $X$  is convergent in  $X$ .

The following definitions are given in [21].

### 2.6 Definition (Measurable function)

Let  $(\Omega, \Sigma)$  be a measurable space with  $\Sigma$ —a sigma algebra of subsets of  $\Omega$  and  $M$  be a nonempty subset of a metric space  $X = (X, d)$ . Let  $2^M$  be the family of nonempty subsets of  $M$  and  $C(M)$  the family of all nonempty closed subsets of  $M$ . A mapping  $G : \Omega \rightarrow 2^M$  is called measurable if for each open subset  $U$  of  $M$ ,  $G^{-1}(U) \in \Sigma$ , where  $G^{-1}(U) = \{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\}$ .

### 2.7 Definition (Measurable selector)

A mapping  $\xi : \Omega \rightarrow M$  is called measurable selector of a measurable mappings  $G : \Omega \rightarrow 2^M$  if  $\xi$  is measurable and  $\xi(\omega) \in G(\omega)$  for each  $\omega \in \Omega$ .

### 2.8 Definition (Random operator)

The mapping  $T : \Omega \times M \rightarrow X$  is called a random operator iff for each fixed  $x \in M$ , the mapping  $T(., x) : \Omega \rightarrow X$  is measurable.

### 2.9 Definition (Continuous random mapping)

A random operator  $T : \Omega \times M \rightarrow X$  is called continuous random operator if for each fixed  $x \in M$  and  $\omega \in \Omega$ , the mapping  $T(\omega, .) : \Omega \rightarrow X$  is continuous.

### 2.10 Definition (Random fixed point)

A measurable mappings  $\xi : \Omega \rightarrow M$  is a random fixed point of a random operator  $T : \Omega \times M \rightarrow X$  iff  $T(\omega, \xi(\omega)) = \xi(\omega)$  for each  $\omega \in \Omega$ .

### 2.11 Definition (Cone random metric space)

Let  $M$  be a nonempty set and the mapping  $d : \Omega \times M \rightarrow p$ , where  $p$  is a cone,  $\omega \in \Omega$  be a selector, satisfy the following conditions:

- (i)  $d(x(\omega), y(\omega)) \geq 0$  and  $d(x(\omega), y(\omega)) = 0 \Leftrightarrow x(\omega) = y(\omega)$  for all  $x(\omega), y(\omega) \in \Omega \times M$ ,
- (ii)  $d(x(\omega), y(\omega)) = d(y(\omega), x(\omega))$  for all  $x, y \in M$ ,  $\omega \in \Omega$  and  $x(\omega), y(\omega) \in \Omega \times M$ ,
- (iii)  $d(x(\omega), y(\omega)) \leq d(x(\omega), z(\omega)) + d(z(\omega), y(\omega))$  for all  $x, y, z \in M$  and  $\omega \in \Omega$  be a selector,
- (iv) for any  $x, y \in M$ ,  $\omega \in \Omega$ ,  $d(x(\omega), y(\omega))$  is non-increasing and left continuous.

Then  $d$  is called cone random metric on  $M$  and  $(M, d)$  is called a cone random metric space.

### 2.12 Definition (Weakly compatible [2])

Random operators  $T, S : \Omega \times X \rightarrow X$  (where  $X$  be a separable Banach space) are weakly compatible if  $T(\omega, S(\omega, \xi(\omega))) = S(\omega, T(\omega, \xi(\omega)))$  provided that  $T(\omega, \xi(\omega)) = S(\omega, \xi(\omega))$  for every  $\omega \in \Omega$ .

Recently, Rashwan and Hammad [17] proved a unique random fixed point in a separable Hilbert space under the following general contractive condition: Let  $T : \Omega \times X \rightarrow X$  be a continuous operator such that for  $\omega \in \Omega$ ,

$$\|T(\omega, x) - T(\omega, y)\| \leq \alpha(\omega) \max \left\{ \begin{array}{l} \|x - y\|, \\ \frac{\beta(\omega)}{2} [\|x - T(\omega, x)\| + \\ \frac{\gamma(\omega)}{2} [\|y - T(\omega, y)\|, \\ \frac{\gamma(\omega)}{2} [\|x - T(\omega, y)\| \\ + \|y - T(\omega, x)\|] \end{array} \right\}, \quad (2.1)$$

for all  $x, y \in X$  where  $\alpha(\omega), \beta(\omega)$  and  $\gamma(\omega)$  are nonnegative real valued random variables such that  $\alpha(\omega), \beta(\omega), \gamma(\omega) \in (0, 1)$  and  $\alpha(\omega) \cdot \beta(\omega) < \alpha(\omega)$ ,  $\alpha(\omega) \cdot \gamma(\omega) < \alpha(\omega)$ .

### 3 Main Results

In this section we shall prove a common random fixed point theorem under a generalized contraction condition for four mappings satisfying some conditions in the setting of cone random metric spaces.

#### 3.1 Theorem

Let  $(X, d)$  be a complete cone random metric space with respect to a cone  $p$  and let  $M$  be a nonempty separable closed subset of  $X$ . Assume that  $S, T, P$  and  $Q$  be four continuous random operators defined on  $M$  such that for  $\omega \in \Omega$ ,  $S(\omega, .), T(\omega, .), P(\omega, .), Q(\omega, .) : \Omega \times M \rightarrow M$  satisfying the following conditions:

- (i)  $S(\omega, X) \subseteq Q(\omega, X)$  and  $T(\omega, X) \subseteq P(\omega, X)$ ,
- (ii) the pairs  $\{S, P\}$  and  $\{T, Q\}$  are weakly random compatible,
- (iii)

$$d(S(x(\omega), T(y(\omega))) \leq \alpha(\omega) \max \left\{ \begin{array}{l} d(P(x(\omega)), Q(y(\omega))), \\ \frac{\beta(\omega)}{2} [d(P(x(\omega)), S(x(\omega))) + d(Q(y(\omega)), T(y(\omega)))] \\ \frac{\gamma(\omega)}{2} [d(P(x(\omega)), T(y(\omega))) + d(Q(y(\omega)), S(x(\omega)))] \end{array} \right\}, \quad (3.1)$$

for all  $x(\omega), y(\omega) \in \Omega \times X$  where  $\alpha(\omega)$ ,  $\beta(\omega)$  and  $\gamma(\omega)$  are nonnegative real valued random variables such that  $\alpha(\omega), \beta(\omega), \gamma(\omega) \in (0, 1)$  and  $\alpha(\omega).\beta(\omega) < \alpha(\omega)$ ,  $\alpha(\omega).\gamma(\omega) < \alpha(\omega)$ . Then the four random mappings have a unique common random fixed point in  $X$ .

**Proof.** For each  $x_0(\omega), x_1(\omega) \in \Omega \times X$  and  $n = 0, 1, 2, \dots$  we choose  $y_1(\omega), y_2(\omega) \in \Omega \times X$  such that  $y_1(\omega) = S(x_0(\omega)) = Q(x_1(\omega))$  and  $y_2(\omega) = T(x_1(\omega)) = P(x_2(\omega))$ . In general we construct a sequence of measurable mappings  $y_n(\omega), x_n(\omega) : \Omega \rightarrow X$  defined by

$$\begin{cases} y_{2n+1}(\omega) = S(x_{2n}(\omega)) = Q(x_{2n+1}(\omega)) \\ y_{2n+2}(\omega) = T(x_{2n+1}(\omega)) = P(x_{2n+2}(\omega)) \end{cases}, \quad (3.2)$$

Then from (3.1) and (3.2), we get

$$\begin{aligned} d(y_{2n+1}(\omega), y_{2n+2}(\omega)) &= d(S(x_{2n}(\omega)), T(x_{2n+1}(\omega))) \\ &\leq \alpha(\omega) \max \left\{ \begin{array}{l} d(P(x_{2n}(\omega)), Q(x_{2n+1}(\omega))), \\ \frac{\beta(\omega)}{2} [d(P(x_{2n}(\omega)), S(x_{2n}(\omega))) \\ + d(Q(x_{2n+1}(\omega)), T(x_{2n+1}(\omega)))] \\ \frac{\gamma(\omega)}{2} [d(P(x_{2n}(\omega)), T(x_{2n+1}(\omega))) \\ + d(Q(x_{2n+1}(\omega)), S(x_{2n}(\omega)))] \end{array} \right\} \\ &\leq \alpha(\omega) \max \left\{ \begin{array}{l} d(y_{2n}(\omega), y_{2n+1}(\omega)), \\ \frac{\beta(\omega)}{2} [d(y_{2n}(\omega), y_{2n+1}(\omega)) + d(y_{2n+1}(\omega), y_{2n+2}(\omega))] \\ \frac{\gamma(\omega)}{2} [d(y_{2n}(\omega), y_{2n+2}(\omega)) + d(y_{2n+1}(\omega), y_{2n+1}(\omega))] \end{array} \right\} \\ &\leq \alpha(\omega) \max \left\{ \begin{array}{l} d(y_{2n}(\omega), y_{2n+1}(\omega)), \\ \frac{\beta(\omega)}{2} [d(y_{2n}(\omega), y_{2n+1}(\omega)) + d(y_{2n+1}(\omega), y_{2n+2}(\omega))] \\ \frac{\gamma(\omega)}{2} [d(y_{2n}(\omega), y_{2n+1}(\omega)) + d(y_{2n+1}(\omega), y_{2n+2}(\omega))] \end{array} \right\}. \end{aligned}$$

For nonnegative real numbers  $a, b$  and  $c$ , if  $\max\{a, b, c\} = a$ , then

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \alpha(\omega)d(y_{2n}(\omega), y_{2n+1}(\omega)).$$

If  $\max\{a, b, c\} = b$ , it follows that

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \frac{\alpha(\omega)\beta(\omega)}{2 - \alpha(\omega)\beta(\omega)} d(y_{2n}(\omega), y_{2n+1}(\omega)) < \frac{\alpha(\omega)}{2 - \alpha(\omega)} d(y_{2n}(\omega), y_{2n+1}(\omega)).$$

If  $\max\{a, b, c\} = c$ , we can write

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \frac{\alpha(\omega)\gamma(\omega)}{2 - \alpha(\omega)\gamma(\omega)} d(y_{2n}(\omega), y_{2n+1}(\omega)) < \frac{\alpha(\omega)}{2 - \alpha(\omega)} d(y_{2n}(\omega), y_{2n+1}(\omega)).$$

Putting

$$\lambda(\omega) = \max\{\alpha(\omega), \frac{\alpha(\omega)}{2 - \alpha(\omega)}\}, \quad (3.3)$$

it's clearly that  $0 < \lambda(\omega) < 1$ .

According to (3.3), we can easy write

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \lambda(\omega)d(y_{2n}(\omega), y_{2n+1}(\omega)).$$

Similarly, we have

$$d(y_{2n}(\omega), y_{2n+1}(\omega)) \leq \lambda(\omega)d(y_{2n-1}(\omega), y_{2n}(\omega)),$$

hence

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \lambda^2(\omega)d(y_{2n-1}(\omega), y_{2n}(\omega)).$$

On continuing this process, we have

$$d(y_{2n+1}(\omega), y_{2n+2}(\omega)) \leq \lambda^{2n}(\omega)d(y_0(\omega), y_1(\omega)).$$

Also, for  $n > m$ , we get

$$\begin{aligned} d(y_n(\omega), y_m(\omega)) &\leq d(y_n(\omega), y_{n-1}(\omega)) + d(y_{n-1}(\omega), y_{n-2}(\omega)) + \dots + d(y_{m+1}(\omega), y_m(\omega)) \\ &\leq (\lambda^{n-1}(\omega) + \lambda^{n-2}(\omega) + \dots + \lambda^m(\omega))d(y_1(\omega), y_0(\omega)) \\ &\leq \left(\frac{\lambda^m(\omega)}{1-\lambda(\omega)}\right)d(y_1(\omega), y_0(\omega)). \end{aligned}$$

Let  $0 \ll \varepsilon$  is given. Choose a natural number  $N$  such that  $\left(\frac{\lambda^m(\omega)}{1-\lambda(\omega)}\right)d(y_1(\omega), y_0(\omega)) \ll \varepsilon$  for every  $m \geq N$ , hence

$$d(y_n(\omega), y_m(\omega)) \leq \left(\frac{\lambda^m(\omega)}{1-\lambda(\omega)}\right)d(y_1(\omega), y_0(\omega)) \ll \varepsilon,$$

this implies that  $\{y_n(\omega)\}$  is a Cauchy sequence in  $\Omega \times X$ .

Since  $(X, d)$  is complete, then there exists  $z(\omega) \in \Omega \times X$  such that  $y_n(\omega) \rightarrow z(\omega)$  as  $n \rightarrow \infty$ . Then from (3.2), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} S(x_{2n}(\omega)) &= \lim_{n \rightarrow \infty} Q(x_{2n+1}(\omega)) = z(\omega) \\ \text{and } \lim_{n \rightarrow \infty} T(x_{2n+1}(\omega)) &= \lim_{n \rightarrow \infty} P(x_{2n+2}(\omega)) = z(\omega), \end{aligned}$$

therefore

$$\lim_{n \rightarrow \infty} S(x_{2n}(\omega)) = \lim_{n \rightarrow \infty} Q(x_{2n+1}(\omega)) = \lim_{n \rightarrow \infty} T(x_{2n+1}(\omega)) = \lim_{n \rightarrow \infty} P(x_{2n+2}(\omega)) = z(\omega). \quad (3.4)$$

Since  $T(\omega, X) \subseteq P(\omega, X)$ , then there exists  $u(\omega) \in \Omega \times X$  such that

$$z(\omega) = P(u(\omega)). \quad (3.5)$$

From (3.1), we obtain

$$\begin{aligned} d(S(u(\omega)), z(\omega)) &\leq d(S(u(\omega)), T(x_{2n+1}(\omega))) + d(T(x_{2n+1}(\omega)), z(\omega)) \\ &\leq \alpha(\omega) \max \left\{ \begin{array}{l} d(P(u(\omega)), Q(x_{2n+1}(\omega))), \\ \frac{\beta(\omega)}{2}[d(P(u(\omega)), S(u(\omega)))] + \\ d(Q(x_{2n+1}(\omega)), T(x_{2n+1}(\omega))), \\ \frac{\gamma(\omega)}{2}[d(P(u(\omega)), T(x_{2n+1}(\omega)))] \\ + d(Q(x_{2n+1}(\omega)), S(u(\omega))) \end{array} \right\} + d(T(x_{2n+1}(\omega)), z(\omega)). \end{aligned}$$

Taking the limit as  $n \rightarrow \infty$  in above inequality and using (3.4) and (3.5), we have

$$\begin{aligned} d(z(\omega), S(u(\omega))) &\leq \alpha(\omega) \max \left\{ 0, \frac{\beta(\omega)}{2}[d(z(\omega), S(u(\omega)))] + \frac{\gamma(\omega)}{2}[d(z(\omega), S(u(\omega)))] \right\} \\ &< \frac{\alpha(\omega)}{2}d(z(\omega), S(u(\omega))), \end{aligned}$$

or,  $(1 - \frac{\alpha(\omega)}{2})d(z(\omega), S(u(\omega))) \leq 0$ , this implies that  $d(z(\omega), S(u(\omega))) \leq 0$ , since  $0 < 1 - \frac{\alpha(\omega)}{2} < 1$ . Thus  $-d(z(\omega), S(u(\omega))) \in p$ . But  $d(z(\omega), S(u(\omega))) \in p$ , therefore by Definition 2.1 (c<sub>3</sub>), we have  $d(z(\omega), S(u(\omega))) = 0$  and so  $z(\omega) = S(u(\omega))$ .

From (3.5) we get

$$z(\omega) = P(u(\omega)) = S(u(\omega)). \quad (3.6)$$

Hence  $u(\omega)$  is a random coincidence point of  $P$  and  $S$ .

Since the pair  $P$  and  $S$  are weakly random compatible, i.e.  $P(S(u(\omega))) = S(P(u(\omega)))$  this implies that

$$P(z(\omega)) = S(z(\omega)). \quad (3.7)$$

Again since  $S(\omega, X) \subseteq Q(\omega, X)$ , then there exists  $v(\omega) \in \Omega \times X$  such that

$$z(\omega) = Q(v(\omega)). \quad (3.8)$$

From (3.1), (3.6) and (3.8), we have

$$\begin{aligned}
 d(z(\omega), T(v(\omega))) &= d(S(u(\omega)), T(v(\omega))) \\
 &\leq \alpha(\omega) \max \left\{ \frac{\beta(\omega)}{2} [d(P(u(\omega)), Q(v(\omega))), d(P(u(\omega)), S(u(\omega))) + d(Q(v(\omega)), T(v(\omega)))], \right. \\
 &\quad \left. \frac{\gamma(\omega)}{2} [d(P(u(\omega)), T(v(\omega))) + d(Q(v(\omega)), S(u(\omega)))] \right\} \\
 &\leq \alpha(\omega) \max \left\{ 0, \frac{\beta(\omega)}{2} [0 + d(z(\omega), T(v(\omega)))] + \frac{\gamma(\omega)}{2} [d(z(\omega), T(v(\omega)))] \right\} \\
 &< \frac{\alpha(\omega)}{2} d(z(\omega), T(v(\omega))),
 \end{aligned}$$

or,  $(1 - \frac{\alpha(\omega)}{2})d(z(\omega), T(v(\omega))) \leq 0$ , this implies that  $d(z(\omega), T(v(\omega))) \leq 0$ , since  $0 < 1 - \lambda(\omega) < 1$ . Thus  $-d(z(\omega), T(v(\omega))) \in p$ . But  $d(z(\omega), T(v(\omega))) \in p$ , therefore by Definition 2.1 (c<sub>3</sub>), we have  $d(z(\omega), T(v(\omega))) = 0$  and so  $z(\omega) = T(v(\omega))$ .

From (3.8) we get

$$z(\omega) = Q(v(\omega)) = T(v(\omega)). \quad (3.9)$$

Hence  $v(\omega)$  is a random coincidence point of  $T$  and  $Q$ .

Since the pair  $T$  and  $Q$  are weakly random compatible, i.e.  $T(Q(v(\omega))) = Q(T(v(\omega)))$  this implies that

$$T(z(\omega)) = Q(z(\omega)). \quad (3.10)$$

Now we show that  $z(\omega)$  is a random fixed point of  $S$ , we have from (3.1) that

$$\begin{aligned}
 d(S(z(\omega)), z(\omega)) &= d(S(z(\omega)), T(v(\omega))) \\
 &\leq \alpha(\omega) \max \left\{ \frac{\beta(\omega)}{2} [d(P(z(\omega)), Q(v(\omega))), d(P(z(\omega)), S(z(\omega))) + d(Q(v(\omega)), T(v(\omega)))], \right. \\
 &\quad \left. \frac{\gamma(\omega)}{2} [d(P(z(\omega)), T(v(\omega))) + d(Q(v(\omega)), S(z(\omega)))] \right\}.
 \end{aligned}$$

Using (3.7) and (3.9), we get

$$\begin{aligned}
 d(S(z(\omega)), z(\omega)) &\leq \alpha(\omega) \max \left\{ \frac{\gamma(\omega)}{2} [d(S(z(\omega)), z(\omega)), 0, d(S(z(\omega)), z(\omega)) + d(z(\omega), S(z(\omega)))] \right\} \\
 &< \frac{\alpha(\omega)}{2} d(S(z(\omega)), z(\omega)),
 \end{aligned}$$

it follows that  $d(S(z(\omega)), z(\omega)) = 0$ , i.e.  $S(z(\omega)) = z(\omega)$ .

According to (3.7), we obtain that

$$P(z(\omega)) = S(z(\omega)) = z(\omega). \quad (3.11)$$

By a similar way and using (3.11), we can prove that for all  $\omega \in \Omega$

$$T(z(\omega)) = Q(z(\omega)) = z(\omega). \quad (3.12)$$

The equations (3.11) and (3.12) shows that  $z(\omega)$  is a common random fixed point of  $T, S, P$  and  $Q$ .

Now, we show the uniqueness. Let  $q(\omega) \neq z(\omega)$  be another common random fixed point of the four mappings, then from (3.1), one can write

$$\begin{aligned}
 d(z(\omega), q(\omega)) &= d(S(z(\omega)), T(q(\omega))) \\
 &\leq \alpha(\omega) \max \left\{ \frac{\beta(\omega)}{2} [d(P(z(\omega)), Q(q(\omega))), d(P(z(\omega)), S(z(\omega))) + d(Q(q(\omega)), T(q(\omega)))], \right. \\
 &\quad \left. \frac{\gamma(\omega)}{2} [d(P(z(\omega)), T(q(\omega))) + d(Q(q(\omega)), S(z(\omega)))] \right\} \\
 &\leq \alpha(\omega) \max \left\{ d(z(\omega), q(\omega)), 0, \frac{\gamma(\omega)}{2} [d(z(\omega), q(\omega)) + d(q(\omega), z(\omega))] \right\} \\
 &< \frac{\alpha(\omega)}{2} d(z(\omega), q(\omega)) < d(z(\omega), q(\omega)),
 \end{aligned}$$

a contradiction. Hence  $q(\omega) = z(\omega)$  and so  $z(\omega)$  is a unique common random fixed point of  $T, S, P$  and  $Q$ . The proof is completed.

If we take  $P = Q$  in above theorem we obtain the following corollary.

### 3.2 Corollary

Let  $(X, d)$  be a complete cone random metric space with respect to a cone  $p$  and let  $M$  be a nonempty separable closed subset of  $X$ . Assume that  $S, T$  and  $P$  are three continuous random operators defined on  $M$  such that for  $\omega \in \Omega$ ,  $S(\omega, .), T(\omega, .), P(\omega, .) : \Omega \times M \rightarrow M$  satisfying the following conditions:

- (i)  $S(\omega, X) \subseteq P(\omega, X)$  and  $T(\omega, X) \subseteq P(\omega, X)$ ,
- (ii) the pairs  $\{S, P\}$  and  $\{T, P\}$  are weakly random compatible,
- (iii)

$$d(S(x(\omega)), T(y(\omega))) \leq \alpha(\omega) \max \left\{ \begin{array}{l} d(P(x(\omega)), P(y(\omega))), \\ \frac{\beta(\omega)}{2} [d(P(x(\omega)), S(x(\omega))) + d(P(y(\omega)), T(y(\omega)))], \\ \frac{\gamma(\omega)}{2} [d(P(x(\omega)), T(y(\omega))) + d(P(y(\omega)), S(x(\omega)))] \end{array} \right\},$$

for all  $x(\omega), y(\omega) \in \Omega \times X$  where  $\alpha(\omega)$ ,  $\beta(\omega)$  and  $\gamma(\omega)$  are nonnegative real valued random variables such that  $\alpha(\omega), \beta(\omega), \gamma(\omega) \in (0, 1)$  and  $\alpha(\omega) \cdot \beta(\omega) < \alpha(\omega)$ ,  $\alpha(\omega) \cdot \gamma(\omega) < \alpha(\omega)$ . Then the three random mappings have a unique common random fixed point in  $X$ .

Putting  $P = Q$  and  $S = T$  in above theorem we get the following corollary.

### 3.3 Corollary

Let  $(X, d)$  be a complete cone random metric space with respect to a cone  $C$  and let  $M$  be a nonempty separable closed subset of  $X$ . Assume that  $S$  and  $P$  are two continuous random operators defined on  $M$  such that for  $\omega \in \Omega$ ,  $S(\omega, .), P(\omega, .) : \Omega \times M \rightarrow M$  satisfying the following conditions:

- (i)  $S(\omega, X) \subseteq P(\omega, X)$ ,
- (ii) the pairs  $\{S, P\}$  is weakly random compatible,
- (iii)

$$d(S(x(\omega)), S(y(\omega))) \leq \alpha(\omega) \max \left\{ \begin{array}{l} d(P(x(\omega)), P(y(\omega))), \\ \frac{\beta(\omega)}{2} [d(P(x(\omega)), S(x(\omega))) + d(P(y(\omega)), S(y(\omega)))], \\ \frac{\gamma(\omega)}{2} [d(P(x(\omega)), S(y(\omega))) + d(P(y(\omega)), S(x(\omega)))] \end{array} \right\},$$

for all  $x(\omega), y(\omega) \in \Omega \times X$  where  $\alpha(\omega)$ ,  $\beta(\omega)$  and  $\gamma(\omega)$  are nonnegative real valued random variables such that  $\alpha(\omega), \beta(\omega), \gamma(\omega) \in (0, 1)$  and  $\alpha(\omega) \cdot \beta(\omega) < \alpha(\omega)$ ,  $\alpha(\omega) \cdot \gamma(\omega) < \alpha(\omega)$ . Then the two random mappings have a unique common random fixed point in  $X$ .

Letting  $P = I$  (where  $I$  is the identity mapping defined by  $I(\omega, x) = x(\omega)$  for all  $\omega \in \Omega$ ) in Corollary 3.3, we have

### 3.4 Corollary

Let  $(X, d)$  be a complete cone random metric space with respect to a cone  $C$  and let  $M$  be a nonempty separable closed subset of  $X$ . Assume that  $S$  be a continuous random operators defined on  $M$  such that for  $\omega \in \Omega$ ,  $S(\omega, .) : \Omega \times M \rightarrow M$  satisfying the condition

$$d(S(x(\omega)), S(y(\omega))) \leq \alpha(\omega) \max \left\{ \begin{array}{l} d(x(\omega), y(\omega)), \\ \frac{\beta(\omega)}{2} [d(x(\omega), S(x(\omega))) + d(y(\omega), S(y(\omega)))], \\ \frac{\gamma(\omega)}{2} [d(x(\omega), S(y(\omega))) + d(y(\omega), S(x(\omega)))] \end{array} \right\},$$

for all  $x(\omega), y(\omega) \in \Omega \times X$  where  $\alpha(\omega)$ ,  $\beta(\omega)$  and  $\gamma(\omega)$  are nonnegative real valued random variables such that  $\alpha(\omega), \beta(\omega), \gamma(\omega) \in (0, 1)$  and  $\alpha(\omega) \cdot \beta(\omega) < \alpha(\omega)$ ,  $\alpha(\omega) \cdot \gamma(\omega) < \alpha(\omega)$ . Then  $S$  has a unique random fixed point in  $X$ .

Finally, we present some examples to verify the requirements of Theorem 3.1 as follows.

### 3.5 Example

Let  $(\Omega, \Sigma)$  denotes a measurable space and  $M = \{1, 2, 3, 4, 5\} \subset \mathbb{R}$  with the usual metric  $d$ . Consider  $\Omega = \{1, 2, 3, 4, 5\}$  and let  $\sum$  be the sigma algebra of Lebesgue's measurable subset of  $\Omega$ . Define  $T, Q, S, P : \Omega \times M \rightarrow M$  by

$$\begin{aligned} S(\omega, x) &= \begin{cases} 3 & \text{if } x=1 \\ 4 & \text{otherwise} \end{cases} \text{ and } Q(\omega, x) = \begin{cases} 5 & \text{if } x=1 \\ 4 & \text{otherwise} \end{cases}, \\ T(\omega, x) &= \begin{cases} 1 & \text{if } x=1 \\ 4 & \text{otherwise} \end{cases} \text{ and } P(\omega, x) = \begin{cases} 2 & \text{if } x=1 \\ 4 & \text{otherwise} \end{cases} \text{ for all } \omega \in \Omega. \end{aligned}$$

Taking measurable sequence  $x_n(\omega) = x(\omega) = 3$ , it's clearly that  $S(\omega, x) \subseteq Q(\omega, x)$  and  $T(\omega, x) \subseteq P(\omega, x)$  and for all  $\omega \in \Omega$ ,  $S(x_n(\omega)) = P(x_n(\omega)) = 4$ ,  $S(P(x_n(\omega))) = P(S(x_n(\omega))) = 4$ , this implies that  $P$  and  $S$  are weakly random compatible mappings, similarly  $T$  and  $Q$  too. To satisfy the condition (1.3), by taking  $x(\omega) = 1$  and  $y(\omega) = 2$ , we can write

$$\begin{aligned} 1 &= d(S(x(\omega)), T(y(\omega))) \leq \alpha(\omega) \max \left\{ \begin{array}{l} d(P(x(\omega)), Q(y(\omega))), \\ \frac{\beta(\omega)}{2} [d(P(x(\omega)), S(x(\omega))) + d(Q(y(\omega)), T(y(\omega)))], \\ \frac{\gamma(\omega)}{2} [d(P(x(\omega)), T(y(\omega))) + d(Q(y(\omega)), S(x(\omega)))] \end{array} \right\} \\ &= \alpha(\omega) \max \left\{ 2, \frac{\beta(\omega)}{2}(1), \frac{\gamma(\omega)}{2}(3) \right\} = 2\alpha(\omega) = 1. \end{aligned}$$

Hence  $\alpha(\omega) = \frac{1}{2} \in (0, 1)$ , therefore all conditions of Theorem 3.1 are satisfied and 4 is a unique random fixed point of  $S, T, P$  and  $Q$ .

### 3.6 Example

Let  $(\Omega, \Sigma)$  be a measurable space and  $M = \{0, 1, 2, 3, 4\} \subset \mathbb{R}$  with the usual metric  $d$ . Consider  $\Omega = \{0, 1, 2, 3, 4\}$  and let  $\Sigma$  be the sigma algebra of Lebesgue's measurable subset of  $\Omega$ . Define  $T, Q, S, P : \Omega \times M \rightarrow M$  for all  $\omega \in \Omega$ , by

$$\begin{aligned} S(\omega, x) &= \begin{cases} 2 & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases} \text{ and } Q(\omega, x) = \begin{cases} 4 & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases}, \\ T(\omega, x) &= \begin{cases} 1 & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases} \text{ and } P(\omega, x) = \begin{cases} 3 & \text{if } x=2 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

By taking measurable sequence  $x_n(\omega) = x(\omega) = 4$ , then it's obvious that  $S(\omega, x) \subseteq Q(\omega, x)$ ,  $T(\omega, x) \subseteq P(\omega, x)$  and for all  $\omega \in \Omega$ ,  $S(x_n(\omega)) = P(x_n(\omega)) = 0$ ,  $S(P(x_n(\omega))) = P(S(x_n(\omega))) = 0$ , this gives  $P$  and  $S$  are weakly random compatible mappings, similarly  $T$  and  $Q$  too. To justify the condition (1.3), by taking  $x(\omega) = 2$  and  $y(\omega) = 3$ , it follows that

$$\begin{aligned} 2 &= d(S(x(\omega)), T(y(\omega))) \leq \alpha(\omega) \max \left\{ \frac{\beta(\omega)}{2} [d(P(x(\omega)), S(x(\omega))) + d(Q(y(\omega)), T(y(\omega)))], \frac{\gamma(\omega)}{2} [d(P(x(\omega)), T(y(\omega))) + d(Q(y(\omega)), S(x(\omega)))] \right\} \\ &= \alpha(\omega) \max \left\{ 3, \frac{\beta(\omega)}{2}(1), \frac{\gamma(\omega)}{2}(5) \right\} = 3\alpha(\omega) = 2. \end{aligned}$$

Hence  $\alpha(\omega) = \frac{2}{3} \in (0, 1)$ , therefore all requirements of Theorem 3.1 are satisfied and 0 is a unique random fixed point of  $S, T, P$  and  $Q$ .

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